APPLICATION OF FUNDAMENTAL LEMMA OF VARIATIONAL CALCULUS TO THE BERNOULLI'S PROBLEM FOR THE SHORTEST TIME

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Abstract

Variational calculus studied methods for finding maximum and minimum values of functional. It has its inception in 1696 year by Johan Bernoulli with its glorious problem: to find a curve, connecting two points *A and B* , which does not lie in a vertical, so that heavy point descending on this curve from position A to reach position *in* for at least time. In functional analysis variational calculus takes the same space, as well as theory of maxima and minimum intensity in the classic analysis .

We will prove a theorem for functional where prove that necessary condition for extreme of functional is the variation of functional is equal to zero. We describe the solution of the equation of Euler with example of application, such as the Bernoulli's problem for the shortest time.

Key words: extreme, functional, condition, curve, time, cycloid.

We will explore for extreme of the functional

$$v[y(x)] = \int_{x_0}^{x_1} F(x, y(x), y'(x))\, dx \quad , \tag{0.1}$$

With the limit points of the allowable set of curves: $y(x_0) = y_0$ and $y(x_1) = y_1$. Will we consider that the function $F(x, y, y')$ is three times differentiable. We know that necessary condition for extreme is the variation in the functional is equal to zero. We will now show how the main theorem is applied to the given functional (0.1).

Assume that extreme reached on two times differentiable curve $y = y(x)$ (required only the existence of a derived from the first line of residue curves, otherwise, it may be that of the curve on which is reached extreme, there is a second derived). We are taking some close to

$y = y(x)$ limit curves $y = \overline{y}(x)$ and include curves $y = y(x)$ and $y = \overline{y}(x)$ to the family curves with one parameter

$$y(x,\alpha) = y(x) + \alpha(\overline{y}(x) - y(x)) \ .$$

When $\alpha = 0$ we receive the curve $y = y(x)$, when $\alpha = 1$ we receive $y = \overline{y}(x)$.

As we already know, the difference $\overline{y}(x) - y(x)$ is called variation of the function $y(x)$ and means with the δy .

The variation δy in variational problems play a role analogous to the role of the increase Δx of an independent variable x in problems for study of extreme of function $f(x)$. The variation of function $\delta y = \overline{y}(x) - y(x)$ is a function of the x .

This function can be differentiated one or several times, as $(\delta y)' = \overline{y}'(x) - y'(x) = \delta y'$ it is generated of the variance is equal to the variance of the generated, and similarly

$$(\delta y)'' = \overline{y}''(x) - y''(x) = \delta y'',$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$(\delta y)^{(k)} = \overline{y}^{(k)}(x) - y^{(k)}(x) = \delta y^{(k)}.$$

And so, we analyze the family $y = y(x,\alpha)$, where $y(x,\alpha) = y(x) + \alpha\delta y$, containing the $\alpha = 0$ curves, of which reaches an extreme, and in some $\alpha = 1$ close tolerances and curves that are called curves of comparison.

If we look at the values of functional (0.1), only of the family curves $y = y(x,\alpha)$, it the functional turned into function of α :

$$v[y(x,\alpha)] = \varphi(\alpha),$$

As in the case that we consider $v[y(x,\alpha)]$ is functional depending on parameter, the value of the parameter α determines the curve of the family $y = y(x,\alpha)$ and so determined and the value of functional $v[y(x,\alpha)]$.

<u>Theorem 1.</u>

If functional $v(y)=\int_{x_0}^{x_1} F(x,y,y')dx$ has a local extreme in y, the necessary condition for extreme of functional is

$$\int_{x_0}^{x_1} [F_y - \frac{d}{dx}F_{y'}]\delta y \, dx = 0, \qquad (0.2)$$

<u>*Proof*</u> <u>*of theorem 1 .*</u>

We analyze the function $\varphi(\alpha)$. It reaches its extreme at $\alpha=0$, and when $\alpha=0$ we receive $y=y(x)$, and the functional, in assumption, reaches extreme compared with any permissible curve, and in particular, in terms of the nearly families curves $y=y(x,\alpha)$.

Necessary condition for extreme of the function $\varphi(\alpha)$ at $\alpha=0$, as is known, is its a derivative is equal to zero at $\alpha=0$, i.e.

$$\varphi'(0)=0 \ .$$

Since

$$\varphi(\alpha)=\int_{x_0}^{x_1} F(x, y(x,\alpha), y_x'(x,\alpha))\, dx,$$

It

$$\varphi'(\alpha)=\int_{x_0}^{x_1}\left[F_y'\frac{\partial}{\partial\alpha}y(x,\alpha)+F_{y'}^{'}\frac{\partial}{\partial\alpha}y'(x,\alpha))\right]dx,$$

Where

$$F_y'=\frac{\partial}{\partial y}F(x,y(x,\alpha),y'(x,\alpha)),$$

$$F_{y'}^{'}=\frac{\partial}{\partial y'}F(x,y(x,\alpha),y'(x,\alpha)),$$

$$\frac{\partial}{\partial\alpha}y(x,\alpha)=\frac{\partial}{\partial\alpha}[y(x)+\alpha\delta y]=\delta y$$

$$\frac{\partial}{\partial\alpha}y'(x,\alpha)=\frac{\partial}{\partial\alpha}[y'(x)+\alpha\delta y']=\delta y',$$

And we get

$$\varphi'(\alpha)=\int_{x_0}^{x_1}\left[F_y(x,y(x,\alpha),y'(x,\alpha))\delta y+F_{y'}(x,y(x,\alpha),y'(x,\alpha))\delta y'\right]dx,$$

$$\varphi'(0)=\int_{x_0}^{x_1}\left[F_y(x,y(x),y'(x))\delta y+F_{y'}(x,y(x),y'(x))\delta y'\right]dx \quad (npu\ \alpha=0).$$

As we know, $\varphi'(0)$ is called variation of functional and means δv.

Necessary condition for extreme of functional is its variation is equal to zero

$$\delta v=0\ .$$

For the functional (0.1) this condition has a type of

$$\int_{x_0}^{x_1}[F_y{}'\delta y+F_{y'}^{'}\delta y']\,dx=0 \tag{0.3}$$

Integrate the equation (0.3) in parts, whereas $\delta y'=(\delta y)'$, we get

$$\delta v = [F'_{y'}\delta y']\Big|_{x_0}^{x_1} + \int_{x_0}^{x_1} [F_y' - \frac{d}{dx}F'_{y'}]\delta y\, dx =$$

$$= \int_{x_0}^{x_1} F'_y \delta y\, dx + F'_{y'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))\delta y(x_1) - F'_{y'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))\delta y(x_0) =$$

$$= \int_{x_0}^{x_1} F'_y \delta y\, dx + F'_{y'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))(\overline{y}(x_1) - y(x_1))$$

$$-F'_{y'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))(\overline{y}(x_0) - y(x_0)) - \int_{x_0}^{x_1} (\delta y) dF'_{y'} =$$

$$= \int_{x_0}^{x_1} F'_y \delta y\, dx + F'_{y'}(x_1, y(x_1,\alpha), y'(x_1,\alpha))(0)$$

$$-F'_{y'}(x_0, y(x_0,\alpha), y'(x_0,\alpha))(0) - \int_{x_0}^{x_1} (\delta y) \frac{d}{dx} F'_{y'}$$

Since, all of the possible (permissible) curves in the given problem pass through fixed limit points, we get

$$\delta v = \int_{x_0}^{x_1} [F_y' - \frac{d}{dx}F'_{y'}]\delta y\, dx \ .$$

□

<u>Note.</u>

The first multiplier $F_y' - \frac{d}{dx}F'_{y'}$ of the curve $y = y(x)$ reaches extreme of the continuous function, and the second multiplier δy, random for the choice of the curve in comparison $y = \overline{y}(x)$, is arbitrary function having passed only certain general conditions, namely: the function δy in the border points $x = x_0$, and $x = x_1$ is equal to zero, continuous and differentiable one or several times, δy or δy and $\delta y'$ are small in absolute value.

To simplified the obtain necessary condition (0.2), we will use the following lemma:

Fundamental lemma of the variational calculus

If for any continuous function $\eta(x)$ is true

$$\int_{x_0}^{x_1} \Phi(x)\eta(x)\,dx = 0,$$

Where the function $\Phi(x)$ is continuous in the interval $[x_0, x_1]$, it

$$\Phi(x) \equiv 0$$

in this interval.

Proof of the fundamental lemma of variational calculus

We accept that, in the point $x = \bar{x}$, resting in the interval (x_0, x_1) , $\Phi(x) \neq 0$, is a contradiction.

Indeed, the continuity of the function $\Phi(x)$, it follows that if $\Phi(\bar{x}) \neq 0$ it $\Phi(x)$ keeps characters in vicinity of $\bar{x}$ ($x_0 \leq x \leq x_1$). We choose function $\eta(x)$ which also retains the mark in that vicinity and is equal to zero outside of this vicinity. We receive

$$\int_{x_0}^{x_1} \Phi(x)\eta(x)\,dx = \int_{\underline{x}_0}^{\bar{x}_1} \Phi(x)\eta(x)\,dx \neq 0,$$

Since product $\Phi(x)\eta(x)$ retains its mark in the interval $x_0 \leq x \leq x_1$ and is equal to zero in the same interval.

And so, we come to a contradiction, therefore $\Phi(x) \equiv 0$.

□

Note .

Adoption of lemma and its proof remain unchanged if the function $\eta(x)$ requires the following restrictions:

$$\eta(x_0) = \eta(x_1) = 0,$$

$\eta(x)$ There is a continuous derived to line n ,

$$\left|\eta^{(s)}(x)\right| < \varepsilon, \quad (s = 0,1,\ldots,q; q \leq n) .$$

The function $\eta(x)$ can be selected, e.g. :

$$\eta(x)=\begin{cases} k(x-\bar{x}_0)^{2n}(x-\bar{x}_1)^{2n}, & x\in[\bar{x}_0,\bar{x}_1] \\ 0 & x\in[x_0,x_1]\setminus[\bar{x}_0,\bar{x}_1] \end{cases},$$

where n is a positive number, k is a constant.

Apparently, that the function $\eta(x)$ satisfies the above conditions: it is a continuous, there is a continuous derived to line $2n-1$, in the points x_0 and x_1 is equal to zero and by reducing the factor by k we can do $\left|\eta^{(s)}(x)\right|<\varepsilon$ for the $\forall x\in[x_0,x_1]$.

Now we will apply the fundamental lemma of variational calculus to simplify the above necessary condition for extreme (0.2) of functional (0.1).

Consequence1.1.

If functional $v(y)=\int_{x_0}^{x_1}F(x,y,y')dx$ reaches extreme of the curve $y=y(x)$, and F_y' and are $\frac{d}{dx}F_{y'}'$ continuous, then it $y=y(x)$ is a solution to the differential equation (equation of Euler)

$$F_y-\frac{d}{dx}F_{y'}=0,$$

Or in an expanded form

$$F_y-F_{xy'}-F_{yy'}y'-F_{y'y'}y''=0 \ .$$

Proof of consequence 1.1 .

The proof of consequence 1.1 follows immediately from the fundamental lemma of variational calculus.

□

This equation is called equation of Euler (1744 year). Integral curve $y=y(x,C_1,C_2)$ equation of Euler is called extreme.

To find a curve, which is reached extreme of functional (0.1) we integrate the equation of Euler and spell out random constants, satisfying the general solution of this equation, of the conditions of borders $y(x_0) = y_0, y(x_1) = y_1$.

Only if they are satisfied with these conditions, can be reached extreme of functional.

However, in order to determine whether they are really extreme (maximum or minimum), must be studied and sufficient conditions for extreme.

To recall, that border problem

$$F_y{'} - \frac{d}{dx} F_{y'}^{'} = 0, \quad y(x_0) = y_0, \quad y(x_1) = y_1 ,$$

not always has a solution, and if there is a solution, then this may not be sole.

It should be taken into account that in many variational problems the existence of solutions is evident, from physical or geometrical sense of the problem, and in the solution of the equations of Euler satisfying the border conditions, only a single extreme may be the solution of the given problem.

Bernoulli 's problem for the shortest time

Two points which are at different distances from the ground and not in a vertical line must be connected to a curve such as

$$y = y(x),$$

that body under the influence of gravitational forces pass in the shortest possible time from the upper to the lower point. We are going to calculate this curve and show that this corresponds to the variation task:

$$\int_0^a \frac{\sqrt{1+y'^2}}{\sqrt{-y}} dx = \min, \quad y(0) = 0, \quad y(a) = -h$$

Solution:

Time to passage from a point p_1 which is higher than that, which is less than p_2 is given by the integral

$$t_{12} = \int_{p_1}^{p_2} \frac{ds}{v} ,$$

where s is the arc length and v is speed. Velocity at each point may be obtained by applying the principle of converting kinetic energy into potential gravitational energy.

$$\Rightarrow \quad \frac{1}{2} mv^2 = mgy ,$$

from which we get $v = \sqrt{2gy}$.

From the equality

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (y')^2}\, dx$$

and substituting in the integral we get

$$t_{1,2} = \int_{p_1}^{p_2} \frac{\sqrt{1 + (y')^2}}{\sqrt{2gy}}\, dx =$$

$$= \int_{p_1}^{p_2} \sqrt{\frac{1 + (y')^2}{2gy}}\, dx$$

$$\Rightarrow \quad f = (1 + y'^2)^{1/2} (2gy)^{-1/2}$$

Now we are applying Euler equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'} \right) = 0$$

From $\frac{\partial f}{\partial x} = 0$, we can use the equality of Bernoulli

$$f - y' \frac{\partial f}{\partial y'} = const$$

$$\Rightarrow \quad \frac{\partial f}{\partial y'} = y'(1+y'^2)^{-1/2}(2gy)^{-1/2}$$

$$\Rightarrow \quad (1+y'^2)^{-1/2}(2gy)^{-1/2} - y'.y'(1+y'^2)^{-1/2}(2gy)^{-1/2} = c$$

$$\Rightarrow \quad (2gy)^{-1/2}\left[(1+y'^2)^{-1/2} - y'^2(1+y'^2)^{-1/2}\right] = c$$

$$\Rightarrow \quad \frac{\left[(1+y'^2)^{-1/2} - y'^2(1+y'^2)^{-1/2}\right](1+y'^2)^{-1/2}}{\sqrt{2gy}\sqrt{1+y'^2}} = c$$

$$\Rightarrow \quad \frac{(1+y'^2)^1 - y'^2(1+y'^2)^0}{\sqrt{2gy}\sqrt{1+y'^2}} = c$$

$$\Rightarrow \quad \frac{1}{\sqrt{2gy}\sqrt{1+y'^2}} = c$$

$$\Rightarrow \quad \frac{1}{\sqrt{2gy}.c} = \sqrt{1+y'^2} \qquad \Big|^2$$

$$\Rightarrow \quad \frac{1}{2gy.c^2} = 1+y'^2$$

$$\Rightarrow \quad \left[1+\left[\frac{dy}{dx}\right]^2\right]y = \frac{1}{2gy.c^2}, \qquad k^2 = 2gy.c^2 \qquad (0.1)$$

Therefore, thesolution of (0.4) is a cycloid

$$x = \frac{1}{2}k^2(\theta - \sin\theta)$$

$$y = \frac{1}{2}k^2(1 - \cos\theta)$$

Conclusion

It should be taken into account that in many variational problems the existence of solutions is evident, from physical or geometrical sense of the problem, and in the solution of the equations of Euler satisfying the border conditions, only a single extreme may be the solution of the given problem.

References:

[1] Assoc. Dr. B. Златанов - metric spaces;

[2] L.Э.Эльсгольц - ДИФФЕРЕНЦИАЛЬНЫЕ equations AND ВАРИАЦИОННОЕ ИСЧИСЛЕНИЕ;

[3] Russak I.B. - Calculus of variations (MA4311, 1996);

[4] brunt B. - The calculus of variations (ISBN 0387402470), Springer, 2004);

[5] Byerly W. - introduction to the calculus of variations (Cambridge, 1917);